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On Rayleigh's investigation of crispations of fluid resting on a vibrating support

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Rayleigh (1883) observed that the frequency of parametrically excited capillary–gravity waves in a container of lateral dimensions large compared with the wavelength was only 75% of the frequency ω_0 calculated from Kelvin's dispersion relation for waves of the observed length, and attributed this discrepancy to friction. A boundary-layer calculation on the assumption that the free surface acts as an inextensible film (as is typical for water in the laboratory) yields a 10% reduction from ω_0. The remaining discrepancy may be plausibly attributed to a contamination-induced reduction of surface tension from the value assumed by Rayleigh, but the possibility remains that nonlinearity could account for a significant shift of the frequency from ω_0. The solution of the weakly nonlinear problem for parametrically excited capillary–gravity waves reveals that this shift is positive for Rayleigh's data, whence the surface tension must have been even smaller than that inferred from Kelvin's dispersion relation. This solution also suggests quantitative errors in the solutions of Ezerskii *et al.* (1986) and Milner (1991) for the limiting case of deep-water capillary waves.

1. Introduction

More than one-hundred years ago Lord Rayleigh (1883) repeated Faraday's (1831) experiments on capillary–gravity waves on the surface of a liquid subjected to a vertical oscillation and confirmed Faraday's conclusion (which had been questioned by Matthiessen) that the frequency of the waves is one-half that of the excitation. Rayleigh's container was a rectangular glass plate mounted on a vibrating iron bar; the driving frequency was $2\omega/2\pi = 31$ Hz, the mean depth of the water was $d = 0.0681$ cm, and the observed wavelength was $2\pi/k = 0.848$ cm. (Rayleigh does not give the lateral dimensions of his glass plate, but his discussion implies that they were large compared with $1/k$, and he does refer to related experiments with a 'shallow pool of mercury 3″ or 4″ in diameter.' Letting $2R = 9$ cm and $2\pi/k = 0.848$ cm, we have $kR = 33$, which seems amply large to justify the neglect of lateral boundaries. However, contact-line effects may be significant for some configurations; see Douady (1990).) The corresponding natural frequency, as calculated by Rayleigh from Kelvin's dispersion relation

$$\omega_0^2 = (gk + \hat{T}k^3)\tanh kd \tag{1}$$

($\hat{T} \equiv T/\rho$ is the kinematic surface tension) using $T = 74$ dyn/cm, was $\omega_0/2\pi = 20.8$ Hz. Rayleigh remarks that 'This should have been 15.5; and the discrepancy is probably to be attributed to friction, whose influence must be to diminish the efficient depth and may easily rise to importance when the total depth is so small.' Whatever the cause, discrepancies of this order are rare in Rayleigh's work and, on that ground alone, merit further investigation.

The assumptions that friction is concentrated in thin (compared with $1/k$) boundary layers at the bottom and free surface and that the free surface acts as an inextensible film (as is typical for water) yield, through the extension to finite depth of Lamb's (1932, §351) calculation,

$$\omega = \omega_0(1-\delta) \equiv \omega_1, \quad \delta \equiv \tfrac{1}{4}k\,(2\nu/\omega)^{\frac{1}{2}} \coth kd(1+\operatorname{sech}^2 kd) \quad (\delta \ll 1), \tag{2a, b}$$

for the frequency of damped free waves, where ω_0 is given by (1), δ is the damping ratio, and ν is the kinematic viscosity. Rayleigh's data and $\nu = 0.010\ \text{cm}^2/\text{s}$ imply $\delta = 0.102$; the corresponding boundary-layer thickness is $(2\nu/\omega)^{\frac{1}{2}} = 0.014$ cm. The ratio $\omega_1/\omega_0 = 1-\delta = 0.90$ compares with the observed value of $\omega/\omega_0 = 0.75$, whence Rayleigh's hypothesis appears to be inadequate for the explanation of the observed discrepancy.

Perhaps the simplest explanation for the remaining discrepancy is that the surface tension was lower than the 74 dyn/cm assumed by Rayleigh in his calculation of ω_0. The notional equality $\omega_0(1-\delta) = \omega$ yields $T = 45$ dyn/cm, and it is clear from Rayleigh's subsequent (1890) work that contamination could account for a reduction of this order.† (Douady 1990 infers $T = 29.5$ dyn/cm from his measurements of capillary waves on 'polluted water', and D. Henderson (private communication) infers $T = 48$ dyn/cm from her measurements of capillary waves on 'tap water.') Still, there remains the possibility that some part of the frequency shift could be associated with nonlinearity, and I therefore proceed to calculate the amplitude-induced frequency shift for laterally unbounded, standing waves with square symmetry (as observed by both Faraday and Rayleigh), driven by the vertical displacement $a_0 \cos 2\omega t, ka_0 \ll 1$.

Following Miles & Henderson (1990), I pose the free-surface displacement in the reference frame of the moving container in the form

$$\eta(\boldsymbol{x},t) = \sum_n \eta_n(t)\, \Psi_n(\boldsymbol{x}; k_n), \tag{3}$$

where the Ψ_n constitute a complete set of normal modes, k_n are the corresponding wavenumbers (the eigenvalues), and η_n are the corresponding generalized coordinates. The free-surface displacement observed by Rayleigh (1883) comprised 'two sets of stationary vibrations superposed, the ridges and furrows of the two sets being perpendicular to one another, and usually parallel to the edges of the (rectangular) plate.' The corresponding pattern of squares is described by the normal mode

$$\Psi_1 = \cos kx + \cos ky \quad (k_1 = k), \tag{4}$$

where x and y are Rayleigh's Cartesian coordinates.

The complete set of normal modes comprises the products of $\cos lkx$ or $\sin lkx$ and $\cos mky$ or $\sin mky$, where l and m are integers. The participating members are determined by the requirement that the correlation coefficient $\langle \Psi_1^2 \Psi n \rangle$, which measures the nonlinear coupling between the primary mode Ψ_1 and the secondary

† Rayleigh cites his 1883 paper in his 1890 paper but does not mention the possibility that contamination could account for the discrepancy that he had earlier attributed to friction. In *The Theory of Sound* (1945, §354), he says (in reference to his 1883 experiments) only that 'The violence of the vibrations and the small depth of the liquid interfere with the accurate calculation of frequency on the basis of the observed wavelength.'

mode Ψ_n, differ from zero (but $\Psi_0 \equiv 1$ is excluded by conservation of mass). Choosing $n \equiv k_n^2/k^2$, we obtain

$$\Psi_2 = 2\cos kx \cos ky \quad (k_2 = \surd 2k), \quad \Psi_4 = \cos 2kx + \cos 2ky \quad (k_4 = 2k). \quad (5a, b)$$

Proceeding as in §§2 and 3 of Miles & Henderson (1990) and posing

$$\eta_1 = a\cos(\omega t + \phi) \quad (ka \ll 1), \quad (6)$$

where a and ϕ are the amplitude and phase of the primary mode, I obtain

$$\omega^2/\omega_1^2 = 1 - \tfrac{1}{2}A(ka \coth kd)^2, \quad (7)$$

where

$$A = A(kd, kl_*) = \tfrac{1}{4} + \tfrac{1}{2}S - \tfrac{1}{2}T^2 + \frac{15}{16}\left[\frac{\boldsymbol{k}^2}{1+\boldsymbol{k}^2}\right]T^2 + \tfrac{1}{2}\kappa_2^{-1}T^4 + \tfrac{1}{8}\kappa_4^{-1}(1+T^2)^2$$
$$-\tfrac{1}{4}(2S-\kappa_2)^{-1}(2S-3T^2)^2 - \tfrac{1}{16}(1+T^2-\kappa_4)^{-1}(3-T^2)^2, \quad (8)$$

$$S \equiv \frac{\surd 2 \tanh kd}{\tanh \surd 2kd}, \quad T \equiv \tanh kd, \quad \boldsymbol{k} = k(\hat{T}/g)^{\frac{1}{2}} \equiv kl_*, \quad \kappa_n \equiv \frac{1+n\boldsymbol{k}^2}{1+\boldsymbol{k}^2} \quad (n = 2, 4). \quad (9a–d)$$

The hypothesis that the primary mode dominates the secondary modes fails in the neighbourhood of $\kappa_4 = 1 + T^2$ owing to the resonance between modes 1 and 4 ($k_4 = 2k_1$ and $\omega_4 = 2\omega_1$, corresponding to Wilton's ripples), which is possible only if $\boldsymbol{k} < 1/\surd 2$. The denominator $2S - \kappa_2$ is positive-definite, whence resonance between modes 1 and 2 is impossible.

Rayleigh's data yield $kd = 0.50$ and $kl_* = 2.00/1.56$ for $\rho T = 74/45$ dyn/cm, for which $A = -0.49/-0.09$. It then follows from (7) that, since $A < 0$, nonlinearity cannot account for $\omega/\omega_1 < 1$ in Rayleigh's experiments for the assumed range of kl_*.

The parameter A is relevant for any weakly nonlinear analysis of laterally unbounded capillary–gravity waves. Its presumable counterpart for deep-water capillary waves ($kd, kl_* \gg 1$) has been calculated by Ezerskii *et al.* (1986) and Milner (1991), but their results, $A = (F + 2R + S + T)/\omega k^2 = 1.38$ and $A = T/4\omega k^2 = 2.95$ in their respective notations, differ from one another and from the limiting value $A = 1.89$ given by (8). I have been unable to resolve these discrepancies, although they appear to reflect computational errors rather than basic differences.

A comment on the square pattern

Faraday (1831) also observed a square pattern and remarks that 'The hexagon, the square, and the equilateral triangle are the only figures that can fill an area perfectly. [Moreover,] the square and the triangle are the only figures that can allow for one half oscillating symmetrically with the other,...and of these two the boundary lines between squares are of shorter extent than those between equilateral triangles of equal area. It is evident therefore that one of these two will be finally assumed, and that this will be the square arrangement; because then the fluid will offer the least resistance in its undulations to the motions of the plate....' The meaning of 'resistance' in this context is unclear to me; however, the precedence of the square may be regarded as a consequence of the resonant interaction

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4 = 0, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4, \quad (10a, b)$$

where the $\boldsymbol{k}_n$ are of equal magnitude k and inclined at $\frac{1}{2}(n-1)\pi$ to the x-axis ($\rightarrow$, $\uparrow$, $\leftarrow$, $\downarrow$ for n = 1,2,3,4), and $\omega_n = \omega$. In contrast, a resonant triad among waves of equal frequency is impossible.

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